Tutorial 3: Selected problems of Assignment 3

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Q1) (Ex 3, Q1) Let $f:[-\pi,\pi]\to\mathbb{C}$ be 2π-periodic integrable.

Suppose f is Hölder continuous at $x\in[-\pi,\pi]$, i.e.

there exists $\alpha\in(0,1), L, \delta_0>0$ such that for all $y\in[-\pi,\pi]$ with $|y-x|<\delta_0$,

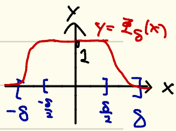
$|f(y)-f(x)|\leq L|y-x|^\alpha$ Show that Conclusion of Theorem 1.5 in Chapter 1

holds: $\lim\limits_{n\to\infty} S_nf(x)=f(x)$, where $S_nf(x):=a_0+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)$

Sol) Modifying the proof of Thm 1.5, using the same notations there:

Fix $n\in\mathbb{N}, \delta>0$: $S_nf(x)-f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_\delta(z)\,\frac{\sin(n+\frac{1}{2})z}{\sin\frac{z}{2}}\,(f(x+z)-f(x))\,dz$

$+\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\Phi_\delta(z))\,\frac{\sin(n+\frac{1}{2})z}{\sin\frac{z}{2}}\cdot(f(x+z)-f(x))\,dz = \mathrm{I}+\mathrm{II}$



For I, by Hölder continuity, $|f(x+z)-f(x)|\leq L|z|^\alpha$, $\forall\ |z|<\delta_0$

Also, $\exists\ \delta_1>0$ s.t. $\left|\frac{\sin z/2}{z/2}\right|>\frac{1}{2}$, $\forall\ |z|<\delta_1$

$\therefore \forall\ |z|<\delta<\min\{\delta_0,\delta_1\}$, $\left|\frac{f(x+z)-f(x)}{\sin z/2}\right|=\frac{|f(x+z)-f(x)|}{|z|^\alpha}\cdot\frac{|z|}{|\sin\frac{z}{2}|}\cdot|z|^{\alpha-1}\leq L\cdot 4\cdot\delta^{\alpha-1}$

$\therefore |\mathrm{I}|\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}|\Phi_\delta(z)|\left|\sin\frac{(n+1)z}{2}\right|\cdot\left|\frac{f(x+z)-f(x)}{\sin z/2}\right|dz\leq\frac{1}{2\pi}\cdot 2\delta(4L\delta^{\alpha-1})=\frac{4\delta^\alpha L}{\pi}$.

$\therefore \forall\ \varepsilon>0$, choose $0<\delta<\min\{\delta_0,\delta_1\}$ such that $\frac{4\delta^\alpha L}{\pi}<\frac{\varepsilon}{2}$

Then the remaining arguments are identical to the original proof.

Q2) (Ex 3, Q2) Let $f:(a,b)\to\mathbb{C}$ be defined. Fix $x_0\in(a,b)$.

a) Suppose f has left and right derivative at x_0.

Show that f is Lipschitz continuous at x_0.

b) Show that converse of (a) does not hold.

Sol) (a) By assumption, set $\varepsilon=1$, there exists $\delta_+,\delta_->0$ such that

$$\begin{cases} \left|\dfrac{f(x_0+z)-f(x_0)}{z}-f'_+(x_0)\right|<1, & \text{for all } 0<z<\delta_+ \\ \left|\dfrac{f(x_0-z)-f(x_0)}{z}-f'_-(x_0)\right|<1, & \text{for all } 0<z<\delta_- \end{cases}$$

Let $M=\max\{|f'_+(x_0)|,|f'_-(x_0)|\}$, $\delta=\min\{\delta_+,\delta_-\}$, then for all $|z|<\delta$,

$|f(x_0+z)-f(x_0)|\leq |z|(1+M)=L|z|$ by letting $L=1+M>0$

$\therefore$ f is Lipschitz continuous at x_0.

(b) Consider $f:[-\pi,\pi]\to\mathbb{R}$ defined as $f(x)=\begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0\end{cases}$

then for all $0\neq x\in[-\pi,\pi]$, $|f(x)-f(0)|=|x||\sin\frac{1}{x}|\leq|x|$. Also $f(0)=0$.

$\therefore$ f is Lipschitz continuous at 0.

However, $f'_\pm(0)=\lim\limits_{x\to 0^\pm}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to 0^\pm}\sin\frac{1}{x}$ do not exist

Q3) [Ex 3, Q3) Let $f:(a,b]\to\mathbb{C}$ be improperly integrable,

i.e. $\forall\, c\in(a,b]$, $f|_{[c,b]}$ is integrable, and $\lim_{c\to a^+}\int_c^b |f|$ exists.

a) Show that if $f(a)$ is well-defined such that $f:[a,b]\to\mathbb{C}$ is integrable,

then $\int_a^b f = \lim_{c\to a^+}\int_c^b f$.

b) Show that Riemann-Lebesgue lemma holds: $\lim_{n\to\infty}\lim_{c\to a^+}\int_c^b f(x)e^{-inx}dx = 0$.

Sol) a) Since f is integrable on $[a,b]$, f is bounded, i.e.

there exists $M>0$ such that for all $x\in[a,b]$, $|f(x)|\leq M$

$\therefore$ Fix $\varepsilon>0$, $\delta>0$ to be determined, then $\forall\, 0<c-a<\delta$,

$\left|\int_c^b f - \int_a^b f\right| = \left|\int_a^c f\right| \leq \int_a^c |f| \leq (c-a)M < \delta M < \varepsilon$ by choosing $\delta = \frac{\varepsilon}{1+M}$.

b) Fix $\varepsilon>0$, Fix $a<c'<b$ such that $(c'-a)M<\frac{\varepsilon}{2}$

By Riemann-Lebesgue lemma on $f|_{[c',b]}$, $\exists\, n_0\in\mathbb{N}$ s.t. $\forall\, n\geq n_0$,

$\left|\int_{c'}^b f(x)e^{-inx}dx\right| < \frac{\varepsilon}{2}$. $\therefore$ Fix $0<\delta<c'-a$. $\forall\, n\geq n_0$, $\forall\, a<c<a+\delta<c'$

$$\left|\int_c^b f(x)e^{-inx}dx\right| \leq \left|\int_{c'}^b f(x)e^{-inx}dx\right| + \int_c^{c'}|f| < \frac{\varepsilon}{2} + (c'-a)M < \varepsilon.$$

$\therefore \lim_{n\to\infty}\lim_{c\to a^+}\int_c^b f(x)e^{-inx}dx = 0$